Wo Recall

More DGA: derived $\otimes$-Hom adjunctions

- k field (or any base comm. ring)

$Ch(k)$: closed symm. mon. cat of chain cplxes of k-modules

$-\otimes-$, $Hom(-,-)$, unit k

Have defined internally to $Ch(k)$ ("diagrammatically"):

dga, comm.-dga, left/right dg-modules, morphisms of such

($\rightsquigarrow$ we hide all the gradings & signs!..)

- $DGM(A)$: the cat of left dg-modules over the dga A

$\downarrow$ quotient Hom's by htpy eq.

$\mathcal{H}(A)$: the htpy cat., not abelian (unless semi-simple!)

but Δed: $M \xrightarrow{f} N \to C \to M[1]$

$\downarrow$ Verdier quot. (a loc. compat. with Δs)

loc.

$\mathcal{D}(A) := \mathcal{H}(A)/\underbrace{\mathcal{H}_{ac}(A)}_{} = \mathcal{H}(A)[\{q\text{-isos}\}^{-1}] = DGM(A)[\{q\text{-isos}\}^{-1}]$

$\llcorner$ thick subcat. of acyclics (M s.t. $H^*(M)$)

- Ex: $A = A^0 \rightsquigarrow DGM(A) = Ch(A)$, e.g.: $A = k$.

- How to compute in $\mathcal{D}(A)$? Use htpy projectives ("K-proj."):

$\mathcal{H}_{pr}(A) := Loc(A) \subseteq \mathcal{H}(A)$, the smallest Δed subcat. cont. A & closed under $\bigoplus_\infty$ ($\overset{\text{(Lemma)}}{\Longrightarrow}$ thick!)

- Theorem:

(1) If $P \in \mathcal{H}_{pr}(A)$, $E \in \mathcal{H}_{ac}(A) \Rightarrow Hom_{H(A)}(P, E) = 0$.

(2) $\forall M \in \mathcal{H}(A)$, $\exists\ \Delta$: $\underset{\mathcal{H}_{pr}}{\underset{\cap}{pM}} \xrightarrow[\underset{q\text{-iso}}{\sim}]{\alpha_M} M \longrightarrow \underset{\mathcal{H}_{ac}}{\underset{\cap}{aM}} \longrightarrow pM[1]$

• **Cor**:

$$\begin{tikzcd} H_{pr}(A) \arrow[r, hook, shift left] \arrow[drr, "\Delta ed\ \simeq"'] & H(A) \arrow[l, shift left, "p(-)", "\perp"'] \arrow[d, shift left] \\ & & \mathcal{D}(A) \arrow[u, dashed, shift left, "\perp"] \end{tikzcd}$$

$$\mathrm{Hom}_{D(A)}(M,N) \cong \mathrm{Hom}_{H(A)}(pM,N)$$

• **Rmk**: (1) $\Rightarrow$ the Δ in (2) is unique up to unique isom. and functorial in M.

• In practice: can use any $P \xrightarrow[qis]{\simeq} M$, $P \in H_{ac}(A)$

Can recognize from constr: semi-free dg mods are h-proj! (or dir. summands!)

Ex: (If $A = A^0$ $\Rightarrow$ right bdd cplx of proj mods is ho-proj?)

If $A = A^{\leq 0}$ ($= A^{\geq 0}$?); $M \in DGM(A)$: $M^\#$ f.g proj / $A^\#$ $\Rightarrow$ ho-proj.

[Avramov et al, Prop. 4.4]

Derived functors

⚠ nice enough! + additive, $F \circ [1] = [1] \circ F$

$$\begin{tikzcd} F: & DGM(A) \arrow[r] \arrow[d] & DGM(B) \arrow[d] & \\ \rightsquigarrow F: & H(A) \arrow[r] \arrow[d, "\ell_A"'] & H(B) \arrow[d, "\ell_B"] & \text{triangulated} \\ & \mathcal{D}(A) \arrow[r, dashed, "\mathbb{L}F"] \arrow[r, bend right=50, "\forall G"'] \arrow[ur, Rightarrow] & \mathcal{D}(B) & \text{triangulated} \end{tikzcd}$$

If F exact ($F(\text{qiso}) \subseteq \text{qiso}$): $\exists !$

• Not in general! $\rightsquigarrow$ def. the left derived functor of F

$$\mathbb{L}F(-) := \ell_B \circ F \circ p(-) \qquad \alpha := \ell_B F(\alpha)$$

$\exists$ nat eq. $\mathbb{L}F(\ell_A M) = \ell_B F(pM) \longrightarrow \ell_B F(M)$

• **Rmk**: 1) Clearly, $\mathbb{L}F$ is Δed by constr.

2) this $(\mathbb{L}F, \alpha: \mathbb{L}F \circ \ell_A \to \ell_B F)$ is the "best" approx. from the left (right Kan extension)

Bimodules

- Define by diagrams in $Ch(k)$:

$X = {}_B X_A$ is left B-mod $B \otimes X \to X$

— right A — $X \otimes A \to X$

+ the two actions commute!

- obtain an adjunction:

$$\begin{array}{c} DGM(A) \\ X \underset{A}{\otimes} - \uparrow \dashv \downarrow \mathrm{Hom}_B(X, -) \\ DGM(B) \end{array}$$

${}_B(X \otimes_A M)$ as coequalizer then remember the B-action

${}_A \mathrm{Hom}_B({}_B X, {}_B N)$ as equalizer then "remember the A-action

- Nice enough add. functors ⇝ descend to $\mathcal{H}(-)$.
- Want to to derive this adjunction!

$$\begin{array}{c} \mathcal{D}(A) \\ \mathbb{L}(X \underset{A}{\otimes} -) = X \overset{\mathbb{L}}{\underset{A}{\otimes}} - \uparrow \dashv \downarrow \mathbb{R}\mathrm{Hom}_B(X, -) \ ? \\ \mathcal{D}(B) \end{array}$$

|| as before ||

t h

- e.g. can dualize the theorem, & the def of der f.
- Or, can resolve X ho-proj as bimodule ($\equiv B \otimes A^{op}$-dg-module)

as in next talk

- For us, enough to assume ${}_B X \in \mathcal{H}_{pr}(B)$

⇝ $\mathbb{R}\mathrm{Hom}_B(X, -) := \mathrm{Hom}_B({}_B X, -)$

because exact functor

• Why adjoint? Can see from Thm. & Cor. + original adj.:

can trace unit & counit! $\longleftarrow$ $D(A)(M, hN) \cong D(B)(tM, N)$

• Ex: $f : A \to B$ morphism of dga's

$\rightsquigarrow$ ${}_B X_A := {}_B \circlearrowright B \circlearrowleft^f_A$ clearly (semi-)free / B

Usually written $\mathbb{L}f^* = t$, $h = f_*$ = restrict the action!

• <u>When get an equivalence?</u>

In Example: If $f : A \xrightarrow{\sim} B$ is a q-iso $\Rightarrow$ equiv.

Proof: Show unit & counit are quasi-iso $\forall$ obj:

unit

$$\begin{tikzcd}
M \arrow[r, dashed] & f_* \mathbb{L}f^*(M) \arrow[d, equal, "\text{def}"] \\
pM \arrow[u, "\sim"] \arrow[r, "\text{unit}"] & f_* f^*(pM) \arrow[d, equal] \\
A \otimes_A pM \arrow[u, "\cong"] \arrow[r, "f \otimes \mathrm{id}"] & {}_B(B \otimes pM)
\end{tikzcd}$$

} qiso because

counit: similar, $P : \xrightarrow[\alpha]{\sim} f_* N$ in $\mathcal{H}(A)$, $f_*N \simeq N$

$$\begin{tikzcd}
B \otimes^{\mathbb{L}}_A f_* N \arrow[r, dashed] & N \\
B \otimes_A P \arrow[u, equal, "\text{def}"] \arrow[ur, "\varepsilon :="] \arrow[r, "\mathrm{id} \otimes \alpha"'] & B \otimes_A f_* N \arrow[u, "\text{counit}"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes P \cong P \arrow[r, "\alpha", "\sim"'] \arrow[d, "f \otimes \mathrm{id}"', "\sim"] & f_* N \\
B \otimes_A P \arrow[ur, "\varepsilon"] \arrow[r, "\mathrm{id} \otimes \alpha"'] & B \otimes_A f_* N \arrow[u, "\text{counit}"', "b \otimes n \mapsto bn"]
\end{tikzcd}$$

$\Rightarrow$ $\varepsilon \in$ QIS.

More generally ? ${}_B X$

$$\begin{array}{c} \mathcal{D}(B) \\ t = X \otimes^L_A - \Big\uparrow \Big\downarrow \mathrm{Hom}(X,-) = h \\ \mathcal{D}(A) \\ \cup \\ A \end{array}$$

${}_B X_A$ hoproj $/B$

Example (see next talk)

Assume that can show that unit & counit are isos for A and X:

$$\eta_A : A \xrightarrow{\sim} h\underbrace{tA}_{X} \quad , \quad \varepsilon_X : thX \xrightarrow{\sim} X .$$

$\Rightarrow$ iso on $\mathrm{Thick}_A(A)$, isom. on $\mathrm{Thick}_B(X)$

$\Rightarrow$ have an equiv. when restricting to $\mathrm{Thick}_A(A) \underset{h}{\overset{t}{\rightleftarrows}} \mathrm{Thick}_B(X)$.

General easy fact:

lemma: $\begin{cases} F, F' : \mathcal{T} \to \mathcal{S} & \text{two triang. functors} \\ \theta : F \to F' & \text{a nat. transf. s.t.} \end{cases}$

$$\begin{array}{ccc} F\Sigma L & \xrightarrow{\simeq} & \Sigma F L \\ \theta_{\Sigma L}\downarrow & \circlearrowleft & \downarrow \Sigma\theta_L \\ F'\Sigma L & \xrightarrow{\simeq} & \Sigma F' L \end{array}$$

$\Rightarrow$ if θ_M is an iso for $\forall M \in$ a set of thick generators then θ is an iso on all $\mathcal{T}$.

Idea: cones:

$$\begin{array}{ccccccc} FL & \to & FM & \to & FN & \to & \Sigma FL \\ \theta_L \downarrow \cong & & \theta_M \downarrow \cong & & \theta_N \downarrow & & \downarrow \Sigma\theta_L \\ F'L & \to & F'M & \to & F'N & \to & \Sigma F'L \end{array}$$

Δ

$F\Sigma L$, $\theta_{\Sigma L}$, $F'\Sigma L$

$\Sigma\theta_L$ also iso

+ sums ✓

summands ✓

$\square$